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The self-dual Chern-Simons $CP(N)$ models

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Abstract

We study the Chern-Simons $CP(N)$ models with a global $U(1)$ symmetry and found the self-dual models among them. The Bogomolnyi-type bound in these self-dual models is a nontrivial generalization of that in the pure $CP(N)$ models. Our models have quite a rich vacuum and soliton structure and approach the many known gauged self-dual models in some limit.

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The $CP(N)$ models have many interesting structures. In two dimensional spacetime their action is conformally invariant and there exist instantons which are topologically nontrivial [1,2]. They exhibit interesting phenomena common to Yang-Mills theories at four dimensions, like confinement and asymptotic freedom [3]. In three dimensions, the $CP(N)$ models have self-dual solitons whose field configuration and interaction have been studied well [1,4].

On the other hand, there have been considerable studies of the relativistic self-dual gauged Higgs systems during the last few years. The gauge group can be abelian or nonabelian [5–8] with Maxwell or Chern-Simons kinetic terms. The vacuum and soliton structure of the models can exhibit a rich variety. The solitons in these models carry the fractional spin and satisfy the fractional statistics.

Recently some hybrids of the $CP(1)$ model, or the $O(3)$ sigma model and abelian (Maxwell) Chern-Simons Higgs models as been proposed [9–11]. These gauged self-dual $CP(1)$ models have the vacuum and soliton structures which inherits the characteristics of their parent models: There are skymion like solitons similar to those in the $CP(1)$ models, topological and nontopological solitons similar to those in the Chern-Simons Higgs models. In addition, there are solitons in the broken phase, which cannot put into a rotationally symmetric form [10].

In this paper, we generalize the gauged $CP(1)$ case to the gauged $CP(N)$ models. The gauge group can be abelian or nonabelian. The kinetic term for the gauge field is chosen to be the Chern-Simons term for the convenience. The crucial condition for the existence of nontrivial structures beside what we get from the naive $CP(N)$ models is the existence of at least one global $U(1)$ symmetry which commutes with the gauge symmetry. This requirement of a global $U(1)$ symmetry was also a key component in finding the nonabelian self-dual Chern-Simons Higgs models [6]. This global $U(1)$ charge could be a part of a gauged abelian group. This work contrasts the previous works [12] on the gauged $CP(N)$ models, where there is no global $U(1)$ symmetry and so the solitonic structure is identical to that of the pure $CP(N)$ models.

The $CP(N)$ model concerns with the space of a $(N+1)$-dimensional complex vector field $z = (z_1, z_2, ..., z_{N+1})$ of unit length, with the equivalence relation under the overall phase rotation, $z \sim e^{i\alpha} z$. This complex projective space of complex dimension N is equivalent to the coset space $SU(N+1)/U(1) \times SU(N)$. (It is quite straightforward to generalize our consideration here to the general Grassmanian models with the manifold $G(M,N) = U(N)/U(M) \times U(N-M)$.) There have been many studies of the $CP(N)$ models in three dimensions [1–4] whose Lagrangian is given as

$$\mathcal{L}_{CP(N)} = \nabla^\mu \bar{z} \nabla_\mu z, \tag{1}$$

where $\nabla_\mu z = \partial_\mu z - (\bar{z}\partial_\mu z) z$. This theory has a global $SU(N+1)$ symmetry and a local $U(1)$ symmetry which removes the degrees of freedom corresponding to the overall phase of z. The self-dual finite energy configurations are given by $z(\vec{x}) = w/|w|$, where w is a (anti)holomorphic rational function of spatial coordinates $x + iy$. In this theory, there is a conserved topological current,

$$k^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu(\bar{z}\partial_\rho z), \tag{2}$$

whose charge $S = \int d^2 x k^0$ is called the degree and measures the second homotopy class of the mapping $z(t, \vec{x})$ from the space R^2 to the manifold $CP(N)$ when $z(t, \vec{x} = \infty) = z_0$ for a constant vector z_0.

We generalize the Lagrangian (1) by adding the gauge coupling, the Chern-Simons term and the potential energy. The Lagrangian we are to consider reads

$$\mathcal{L} = \frac{\kappa}{2}\epsilon^{\mu\nu\rho}\Big(A^a_\mu \partial_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\Big) + |\nabla_\mu z|^2 - U(z), \tag{3}$$

where the covariant derivative is defined as

$$\nabla_\mu z = D_\mu z - (\bar{z} D_\mu z) z, \tag{4}$$

where $D_\mu z = \partial_\mu z - i A^a_\mu R^a z$. The gauge group is a proper subgroup of $SU(N+1)$, which is generated by the hermitian traceless matrices R^a satisfying the relation $[R^a, R^b] = i f^{abc} R^c$.

We require that there is a conserved global abelian symmetry whose generator is chosen to be $T^D \equiv (1, 1, \ldots, 1, -N)/\sqrt{2N(N+1)}$. If the gauge group is purely nonabelian, the generators R^a for the gauge group should have vanishing elements on the $(N+1)$-th row and column so that the R^a's commute with T^D. If there are more conserved global $U(1)$ symmetries, there are corresponding complicated models, which we not pursued here. The potential $U(z)$ will be specified later by requiring that there is a Bogomolnyi-type bound on the energy.

The kinetic energy for the gauge field can also have the Yang-Mills term. In this case, we need also a neutral scalar field N^a in the adjoint representation of the gauge group for the self-duality. The qualitative behavior does not change much from the pure Chern-Simons case, and so we will not also pursue this direction in this paper.

The theory possesses the following conserved topological current K^μ

$$K^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu(\bar{z}D_\rho z), \tag{5}$$

which is a gauge-invariant generalization of k^μ in Eq. (2). The Gauss law constraint obtained from the variation of A_0^a is

$$\kappa F_{12}^a - i\left\{\nabla_0\bar{z}\left(R^a z - z(\bar{z}R^a z)\right) - \text{h.c.}\right\} = 0. \tag{6}$$

Any physical field configurations should satisfy this constraint. This constraint is preserved in the time evolution and so any physical configuration at a given moment will remain to be physical.

There is also a conserved global $U(1)$ current for the generator T^D,

$$J^\mu = i\left\{\nabla^\mu\bar{z}\left(T^D z - z(\bar{z}T^D z)\right) - \text{h.c.}\right\}. \tag{7}$$

Note that $\nabla_\mu\bar{z}z = 0$, and so some terms in the Gauss law (6) and the global $U(1)$ current (7) vanish. However, we will find later that keeping those terms is useful.

We start with the energy functional

$$E = \int d^2x\left\{|\nabla_0 z|^2 + |\nabla_i z|^2 + U(z)\right\}. \tag{8}$$

With the Gauss law, we get

$$
\begin{aligned}
|\nabla_i z|^2 &= |(\nabla_1 \pm i\nabla_2)z|^2 \pm (K_0 + F^a_{12}(\bar{z}R^a z)) \\
&= |(\nabla_1 \pm i\nabla_2)z|^2 \pm (K_0 + \frac{v}{\kappa}J_0) \\
&\quad \pm \frac{i}{\kappa}\left\{\nabla_0 \bar{z}\left[(R^a z - z(\bar{z}R^a z))(\bar{z}R^a z) - v\left(T^D z - z(\bar{z}T^D z)\right)\right] - \text{h.c.}\right\}, \qquad (9)
\end{aligned}
$$

up to a total derivative which does not contribute to the energy. Here we added and then subtracted a term proportional to the conserved global $U(1)$ charge density. This allows us to introduce a free real parameter v to the model. (If there are more conserved global $U(1)$ current, we may introduce more free parameters like v.)

We choose the potential as

$$
U(z) = \frac{1}{\kappa^2}\left|(R^a z - z(\bar{z}R^a z))\,(\bar{z}R^a z) - v\left(T^D z - z(\bar{z}T^D z)\right)\right|^2 . \qquad (10)
$$

This choice and Eq. (9) allow us to express the energy density as

$$
\begin{aligned}
\mathcal{E} &= |(\nabla_1 \pm i\nabla_2)z|^2 \\
&\quad + \left|\nabla_0 z \pm \frac{i}{\kappa}\left\{(R^a z - z(\bar{z}R^a z))(\bar{z}R^a z) - v\left(T^D z - z(\bar{z}T^D z)\right)\right\}\right|^2 \\
&\quad \pm (K_0 + \frac{v}{\kappa}J_0). \qquad (11)
\end{aligned}
$$

Since the first two terms in the right hand side are nonnegative, there is a Bogomolnyi-type energy bound $E \geq |T|$, where the 'topological charge'

$$
T = \int d^2x (K_0 + \frac{v}{\kappa}J_0) \qquad (12)
$$

is a generalization of the degree S.

The field configurations saturating the energy bound satisfy the Gauss law and the 'self-dual' equations

$$
(\nabla_1 \pm i\nabla_2)z = 0, \qquad (13)
$$

$$
\nabla_0 z \pm \frac{i}{\kappa}\left\{(R^a - \bar{z}R^a z)(\bar{z}R^a z) - v\left(T^D - \bar{z}T^D z\right)\right\} z = 0. \qquad (14)
$$

The reason for keeping the vanishing terms in Eqs. (6) and (7) is obvious now. From Eq. (14) the identity $\bar{z}\nabla_0 z = 0$ does not lead any condition on the z, because the rest of Eq. (14) also vanishes when it is multiplied by $\bar{z}$. This trick does not appear in Ref. [12], resulting in a nontrivial condition on z. Combined with the Gauss law, Eq. (14) gives

$$\begin{aligned} \kappa^2 F^a_{12} = \mp \Big\{ & \Big(\bar{z}(R^a R^b + R^b R^a) z - 2(\bar{z}R^a z)(\bar{z}R^b z) \Big) (\bar{z}R^b z) \\ & -v \Big(\bar{z}(R^a T^D + T^D R^a) z - 2(\bar{z}T^D z)(\bar{z}R^a z) \Big) \Big\} . \end{aligned} \tag{15}$$

If the gauge group were the full $SU(N+1)$, there would be no conserved abelian global current and so $v = 0$. One can show in this case the field strength vanishes. This makes the gauge field part of the self-dual configurations trivial. The potential of this case turns out to be a constant. The soliton structure turns out to be identical to that of the $CP(N)$ model [12].

When Eq. (14) is satisfied, the $U(1)$ charge density from Eq. (7) becomes

$$\begin{aligned} \kappa J_0 = \mp 2 \Big\{ & \Big(\bar{z}(R^a T^D) z - (\bar{z}T^D z)(\bar{z}R^a z) \Big) (\bar{z}R^a z) \\ & -v \Big(\bar{z}(T^D T^D) z - (\bar{z}T^D z)(\bar{z}T^D z) \Big) \Big\} . \end{aligned} \tag{16}$$

For the configuration satisfying Eqs. (13) and (14), the total angular momentum becomes

$$\begin{aligned} J &= -\int d^2x \epsilon_{ij} x^j (\nabla_0 \bar{z} \nabla_j z + \nabla_j \bar{z} \nabla_0 z) \\ &= \frac{1}{2\kappa} \int d^2x x^i \partial_i \Big((\bar{z}R^a z)^2 - 2v(\bar{z}T^D z) \Big) . \end{aligned} \tag{17}$$

To see the $O(3)$ sigma model studied in Ref. [9,10], we consider the $CP(1)$ model with a local abelian gauge symmetry with it generator $T^D = \mathrm{diag}(1,-1)/2$. We can parameterize the configuration space by a unit vector field $\vec{\psi} = \bar{z}\vec{\sigma}z$. With these identification we can construct the following self-dual equations from Eqs. (13) and (14);

$$D_1 \vec{\psi} \pm \vec{\psi} \times D_2 \vec{\psi} = 0, \tag{18}$$

$$D_0 \vec{\psi} = \mp \frac{1}{2\kappa} (2v - \vec{n} \cdot \vec{\psi})(\vec{n} \times \vec{\psi}), \tag{19}$$

with the Gauss law constraint

$$2\kappa F_{12} + \vec{n}\cdot\vec{\psi}\times D_0\vec{\psi} = 0, \tag{20}$$

where $D_\mu\vec{\psi} = \partial_\mu\vec{\psi} + A_\alpha\vec{n}\times\vec{\psi}$ and $\vec{n} = (0,0,1)$. With substituting $v \to v/2$ and $\kappa \to \kappa/2$, we can see that these are the same self-dual equations as the ones considered in Ref. [10].

To understand the general implications of the theory when the gauge group is purely nonabelian, we reparameterize the $N+1$ dimensional vector as $z = e^{i\alpha}(\phi, \sqrt{1-|\phi|^2})$ where ϕ is a N-dimensional complex vector such that $|\phi| \le 1$. As argued before, the gauge generators R^a have the vanishing $N+1$-th column and row, and so the we can define the covariant derivative of ϕ as $D_\mu\phi = \partial_\mu\phi - iA^a_\mu R^a\phi$. The overall phase α has been gauged way. The self-dual equations (13) and (14) for the first N components become

$$(D_1 \pm iD_2)\phi = \frac{1}{2}\left\{\phi^\dagger(D_1\phi \pm iD_2\phi) - (D_1\phi^\dagger \pm iD_2\phi^\dagger)\phi\right\}\phi, \tag{21}$$

$$D_0\phi - \frac{1}{2}(\phi^\dagger D_0\phi - D_0\phi^\dagger\phi)\phi \pm \frac{i}{\kappa}\left\{(R^a - (\phi^\dagger R^a\phi))(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}(1-|\phi|^2)\right\}\phi = 0. \tag{22}$$

The $(N+1)$-th component of the self-dual equations follow from the above equations. With this representation of z, the Gauss law (15) and the charge density (16) also get somewhat simplified.

The potential energy (10) in the pure nonabelian case can be expanded and put together as

$$\begin{aligned} U = {} & \frac{1}{\kappa^2}\left|\Big(R^a\phi - \phi(\phi^\dagger R^a\phi)\Big)(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}\phi\,(1-|\phi|^2)\right|^2 \\ & + \frac{1-|\phi|^2}{\kappa^2}\left|(\phi^\dagger R^a\phi)(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}|\phi|^2\right|^2. \end{aligned} \tag{23}$$

Since the potential is nonnegative, the minimum of the potential could be zero. The ground configuration with the zero potential energy will satisfies the $(N+1)$ conditions,

$$\Big(R^a\phi - \phi(\phi^\dagger R^a\phi)\Big)(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}\phi\,(1-|\phi|^2) = 0, \tag{24}$$

$$\sqrt{1-|\phi|^2}\left\{(\phi^\dagger R^a\phi)(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}|\phi|^2\right\} = 0, \tag{25}$$

not all of which are independent of each other. (Here ϕ denotes the vacuum expectation value of the ϕ field.) One can easily show that F^a_{12} and J^0 for the ground configurations

vanish as expected. When $|\phi| \neq 1$, we can substitute the second equation to the first to get a simplified vacuum equation on the expectation value,

$$R^a\phi(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}\phi = 0, \tag{26}$$

which implies Eqs. (24) and (25). When $|\phi| = 1$, instead we get a different condition

$$\Big(R^a\phi - \phi(\phi^\dagger R^a\phi)\Big)\,(\phi^\dagger R^a\phi) = 0. \tag{27}$$

These equations (26) and (27) determine the vacuum structure of the model.

To understand the vacuum and soliton structure of the theory, let us consider the Higgs limit, $0 < v \ll 1$ and $|\phi| \ll 1$. The kinetic part of the Lagrangian (3) becomes $\nabla_\mu\bar{z}\nabla^\mu z = D_\mu\phi^\dagger D^\mu\phi + O(\phi^4)$. From the potential energy (23) we see that the small ϕ, v limit is consistent if $\phi \sim O(\sqrt{v})$ and that the potential energy in this limit becomes

$$U = \frac{1}{\kappa^2}\left|R^a\phi(\phi^\dagger R^a\phi) - v\sqrt{\tfrac{N+1}{2N}}\phi\right|^2, \tag{28}$$

to order v^4. This is exactly the potential appeared in the self-dual Chern-Simons Higgs models [6]. Thus in this limit, our models become self-dual Chern-Simons Higgs systems. These models have been studied before and shown to have the rich vacuum and solitonic structures [6–8]. As the vacuum condition (26) is identical to that in the the Higgs case, the vacuum structure of the Higgs systems will survive in our models. Thus if $|\phi| < 1$ is satisfied at the vacuum, the vacuum structure of the Higgs limit is identical to that of our model. Especially a rich vacuum structure appears when the matter field is in the adjoint representation of the gauge group [7,8].

Having see that the vacuum structure of the case $|\phi| < 1$, we ask whether there is any nontrivial vacuum structure in the case $|\phi| = 1$. We have analyzed Eq. (27) in detail in the $CP(3)$ case with the ϕ field in the adjoint representation of $SU(2)$. It turns out that Eq. (27) has two class of solutions: one which is a continuation of the solutions of Eq. (26), and another of a different characteristic. It would be interesting to understand the solutions of Eq. (27) in more general situation.

To get another perspective of the models proposed here, we have considered various choices of the gauge group, studied some characteristics of those models, and list them as follows: (I) The simplest case occurs when the theory is not gauged at all. Since Eqs. (9) make sense even if we put $R^a = 0$, our energy bound and self-dual equations still work. Our models in this case can be regarded as the generalization of the self-dual nongauged $O(3)$ model studied by Leese [13]. (II) We can gauge with T^D as its generator. Since the vacuum manifold in the broken phase can be shown to be simply connected for $CP(N)$ with $N \geq 2$, this theory would lead to the semilocal strings in the broken phase [14]. (III) We can gauge the $SU(N)$ subgroup under which the first N components of the complex vector transform as a fundamental representation. There is a global $U(1)$ symmetry generated by T^D. (IV) We can also gauge this global $U(1)$ symmetry, ending up with a gauge group $SU(N) \times U(1)$. (V) For $CP(N^2-1)$, we can gauge $SU(N)$ with the first N^2-1 components belongs to the adjoint representation of the gauge group. This case has a rich vacuum and soliton structure.

In short, we have constructed the self-dual gauged Chern-Simons $CP(N)$ models. The crucial ingredient is that there is at least one global $U(1)$ charge which commutes with the local gauge symmetry. These models generalize nontrivially both the $CP(N)$ models and the self-dual Chern-Simons Higgs models. The vacuum and soliton structure of the models can be quite rich, depending on the parameter v and the gauge symmetry. We have seen that many known self-dual Chern-Simons Higgs models can be obtained by taking a suitable limit.

There are many directions to explore from this point. There may be some new vacuum structures which are not apparent in the parent models. Naturally we expect all self-dual solitons of the parent models to be present in our model and there may be also a new type of solitons as in the $CP(1)$ models. We hope to study these in some concrete model. The models we may obtain from the dimensional reduction to two dimensional space time will be interesting as they are renormalizable. Classically these dimensionally reduced models will have rich domain-wall solitons.

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